PHYS393: Statistical and Low Temperature Physics

Tutorial 3

The following constants may be useful:

Planck's constant,	h	=	6.63×10^{-34} Js
Boltzmann constant,	k_B	=	1.38×10^{-23} J/K
Speed of light,	c	=	3.00×10^{8} m/s
Electron mass,	m_e	=	9.11×10^{-31} kg

1. A system is composed of N conduction electrons that move freely within a cube of metal of side length L.

   (a) Taking account of the spin degeneracy of the electrons, show that the number of states $g(k)\,dk$ with wave number k in the range k to $k+dk$ is:

   $$g(k)\,dk = \frac{2V}{(2\pi)^3}4\pi k^2\,dk,$$

   where the volume $V = L^3$.

   (b) Write down the relationship between wave number k and energy ε. Hence, show that the number of states with energy in the range ε to $\varepsilon + d\varepsilon$ is:

   $$g(\varepsilon)\,d\varepsilon = \frac{2V}{(2\pi)^2}\left(\frac{2m}{\hbar^2}\right)^{\frac{3}{2}}\varepsilon^{\frac{1}{2}}\,d\varepsilon.$$

   (c) The probability that a state with energy ε will be occupied by an electron is $f(\varepsilon)$. Sketch graphs of $f(\varepsilon)$ versus ε: (i) for temperature $T = 0$; (ii) for temperature T such that $0 < T < T_F$, where T_F is the Fermi temperature. Indicate the Fermi energy ε_F.

   (d) Show that the Fermi energy ε_F is given by:

   $$\varepsilon_F = \frac{\hbar^2}{2m}\left(\frac{3\pi^2 N}{V}\right)^{\frac{2}{3}}.$$

   (e) Metallic silver has a molar volume of 10.27×10^{-6} m^3, and each atom contributes one electron to the conduction band. Evaluate the Fermi energy ε_F for silver.

   (f) At temperature T, the total energy U of the electrons can be written:

   $$U = \frac{3}{5}N\varepsilon_F\left[1 + \frac{5\pi^2}{12}\left(\frac{k_B T}{\varepsilon_F}\right)^2\right]$$

   Evaluate the electron heat capacity C_V for a molar quantity of silver at a temperature of 5 K.

2. (a) Explain what is meant by black body radiation.

(b) The density of states $g(k)$ in terms of wave number k for quantised electromagnetic waves in a cavity of volume V is written as:

$$g(k)\,dk = \frac{2V}{(2\pi)^3} 4\pi k^2\,dk.$$

Show that this density of states can be written in terms of the frequency ν of the waves as:

$$g(\nu)\,d\nu = \frac{8\pi V}{c^3}\nu^2\,d\nu,$$

where c is the speed of light.

(c) The energy contained in the frequency interval ν to $\nu + d\nu$ of the radiation is given by:

$$u(\nu)\,d\nu = \frac{8\pi V}{c^3}\nu^2\,d\nu \cdot h\nu \cdot \frac{1}{e^{\frac{h\nu}{k_B T}} - 1}.$$

Explain the physical significance of the factors $h\nu$ and $[\exp(h\nu/k_B T) - 1]^{-1}$ in this expression.

(d) Deduce the limiting values of $u(\nu)$ as $\nu \rightarrow 0$ and as $\nu \rightarrow \infty$.

(e) Sketch the distribution of $u(\nu)$ versus ν.

(f) Show that the energy density U/V in the cavity is given by:

$$\frac{U}{V} = \frac{8\pi^5}{15}\frac{(k_B T)^4}{(hc)^3}.$$

(g) Evaluate the energy density U/V at a temperature $T = 1000$ K.

(h) Calculate the temperature at which the energy density is 10 times greater than it is at $T = 1000$ K.

You are given that:

$$\int_0^\infty \frac{y^3\,dy}{e^y - 1} = \frac{\pi^4}{15}.$$